Monotonicity of relative entropy, Golden-Thompson inequality, Lieb's triple matrix inequality & strong subadditivity of quantum entropy

Zoltán Kolarovszki

March 12, 2020

1 Basics

Theorem 1.1 (Spectral theorem)**.** For any $A \in \mathcal{B}(\mathcal{H})_+$ we have the decomposition

$$A = \sum_{a \in \mathrm{spec}(A)} a P^A(a),$$

where $\{P^A(a)\}_{a \in \mathrm{spec}(A)}$ are orthonormal projections.

Remark. The theorem extends to normal operators. A is normal when $A^*A = AA^*$. One can show that normal operators are unitarily diagonalizable.

Definition 1.1 (von Neumann entropy)**.** Let $\rho \in \mathcal{S}(\mathcal{H})$. The *von Neumann entropy* is defined by

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho).$$

Definition 1.2 (Relative entropy)**.** Let $\rho, \sigma \in \mathcal{S}(\mathcal{H})$. The *relative entropy* is defined by

$$D(\rho \,||\, \sigma) = \mathrm{Tr}(\rho \log \rho - \rho \log \sigma).$$

Definition 1.3 (Completely positive maps)**.** Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ a map. It is called a *completely positive*, if for any $\mathcal{H}_E$

$$\mathrm{id}_{\mathcal{B}(\mathcal{H}_E)} \otimes \Phi : \mathcal{B}(\mathcal{H}_E \otimes \mathcal{H}) \to \mathcal{B}(\mathcal{H}_E \otimes \mathcal{K})$$

is positive.

Definition 1.4 (Trace-preserving maps)**.** Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ a map. It is called a *trace-preserving*, if for any $X \in \mathcal{B}(\mathcal{H})$

$$\mathrm{Tr}\Phi(X) = \mathrm{Tr}X.$$

Maps which are completely positive and trace-preserving are called CPTP maps.

2 Relative entropy monotonicity

2.1 Adjoining a system

Let $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, and $v \in \mathcal{B}(\mathcal{H}_E)$, where $\mathcal{H}_E$ is some auxiliary Hilbert space. We need to show that

$$D(\rho \,||\, \sigma) \geq D\big(\rho \otimes |v\rangle\langle v| \,||\, \sigma \otimes |v\rangle\langle v|\big).$$

.

Actually, it turns out to be an equality. We need to use the following lemma:

Lemma 2.1. Let $\rho, \sigma \in \mathcal{B}(\mathcal{H}_A)$ and $\tau \in \mathcal{B}(\mathcal{H}_B)$, where $\mathcal{H}_A, \mathcal{H}_B$ are Hilbert spaces. Then

$$\log(\rho \otimes \tau) = \log(\rho) \otimes \mathrm{id}_B + \mathrm{id}_A \otimes \log(\tau).$$

Proof. Logarithm on tensor products is just

$$\log(\rho \otimes \tau) = \log\left(\sum_{a\in\mathrm{spec}(\rho)} aP^\rho(a) \otimes \sum_{b\in\mathrm{spec}(\tau)} bP^\tau(b)\right) = \sum_{a\in\mathrm{spec}(\rho)}\sum_{b\in\mathrm{spec}(\tau)} \log(ab)P^\rho(a)\otimes P^\tau(b),$$

where in the last line $\{P^\rho(a)\otimes P^\tau(b)\}_{a\in\mathrm{spec}(\rho),b\in\mathrm{spec}(\tau)}$ are orthonormal projections. Then

$$\begin{aligned}
&\sum_{a\in\mathrm{spec}(\rho)}\sum_{b\in\mathrm{spec}(\tau)} \log(ab)P^\rho(a)\otimes P^\tau(b)\\
&\quad= \sum_{a\in\mathrm{spec}(\rho)}\sum_{b\in\mathrm{spec}(\tau)} \big(\log(a)+\log(b)\big)P^\rho(a)\otimes P^\tau(b)\\
&\quad= \sum_{a\in\mathrm{spec}(\rho)}\sum_{b\in\mathrm{spec}(\tau)} \log(a)P^\rho(a)\otimes P^\tau(b) + \log(b)P^\rho(a)\otimes P^\tau(b)\\
&\quad= \left(\sum_{a\in\mathrm{spec}(\rho)} \log(a)P^\rho(a)\right)\otimes \sum_{b\in\mathrm{spec}(\tau)} P^\tau(b) + \sum_{a\in\mathrm{spec}(\rho)} P^\rho(a)\otimes\left(\sum_{b\in\mathrm{spec}(\tau)} \log(b)P^\tau(b)\right).
\end{aligned}$$

Now we're assuming that ρ,τ have full rank, i. e. $\sum_{a\in\mathrm{spec}(\rho)} P^\rho(a) = \mathrm{id}_A$ and $\otimes\sum_{b\in\mathrm{spec}(\tau)} P^\tau = \mathrm{id}_B$. Therefore we get

$$\left(\sum_{a\in\mathrm{spec}(\rho)} \log(a)P^\rho(a)\right)\otimes \sum_{b\in\mathrm{spec}(\tau)} P^\tau(b) + \sum_{a\in\mathrm{spec}(\rho)} P^\rho(a)\otimes\left(\sum_{b\in\mathrm{spec}(\tau)} \log(b)P^\tau(b)\right)$$

$$= \log(\rho)\otimes\mathrm{id}_B + \mathrm{id}_A\otimes\log(\tau).$$

□

Theorem 2.2. Let $\rho,\sigma\in\mathcal{S}(\mathcal{H}_A)$ and $\tau,\upsilon\in\mathcal{S}(\mathcal{H}_B)$, where $\mathcal{H}_A,\mathcal{H}_B$ are Hilbert spaces. Then

$$D(\rho\otimes\tau\,||\,\sigma\otimes\upsilon) = D(\rho\,||\,\sigma) + D(\tau\,||\,\upsilon).$$

Proof. By direct calculation

$$\begin{aligned}
D(\rho\otimes\tau\,||\,\sigma\otimes\upsilon) &= \mathrm{Tr}\big((\rho\otimes\tau)(\log(\rho\otimes\tau)-\log(\sigma\otimes\upsilon)\big)\\
&= \mathrm{Tr}\big((\rho\otimes\tau)(\log\rho\otimes\mathrm{id}_B + \mathrm{id}_A\otimes\log\tau - \log\sigma\otimes\mathrm{id}_B - \mathrm{id}_A\otimes\log\upsilon)\big)\\
&= \mathrm{Tr}(\rho\log\rho\otimes\tau + \rho\otimes\tau\log\tau - \rho\log\sigma\otimes\tau - \rho\otimes\tau\log\upsilon)\\
&= \mathrm{Tr}(\rho\log\rho - \rho\log\sigma) + \mathrm{Tr}(\tau\log\tau - \tau\log\upsilon) = D(\rho\,||\,\sigma) + D(\tau\,||\,\upsilon).
\end{aligned}$$

□

Corollary 2.2.1.

$$D\big(\rho\otimes\tau\,||\,\sigma\otimes\tau\big) = D(\rho\,||\,\sigma).$$

Proof. By direct calculation using theorem (2.2) we get

$$D\big(\rho\otimes\tau\,||\,\sigma\otimes\tau\big) = D(\rho\,||\,\sigma) + D(\tau\,||\,\tau) = D(\rho\,||\,\sigma) + 0 = D(\rho\,||\,\sigma),$$

where $D(\tau\,||\,\tau) = 0$ follows from the definition (1.2). □

Remark. Just some physical intuition: if you have a system in the laboratory, and another auxiliary system in e. g. the Moon, which are combined but independent (i. e. a product state), then the physical quantities (in this case entropy) in the laboratory does not depend on the auxiliary system. It makes physical sense: adjoining an independent system should not change anything in the measurements. But if you evolve the laboratory and auxiliary system together, then the relative entropy can decrease.

2.2 Unitary evolution

Let $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, and $U \in \mathcal{B}(\mathcal{H})$ unitary operator. We need to show that

$$D(\rho \,||\, \sigma) \geq D(U\rho U^* \,||\, U\sigma U^*).$$

Lemma 2.3. Let $\rho \in \mathcal{B}(\mathcal{H})$ and U unitary operators. Then

$$\log(U\rho U^*) = U \log(\rho) U^*$$

Proof. Using the spectral decomposition we get (1.1)

$$U \log(\rho) U^* = U \sum_{a \in \mathrm{spec}(\rho)} \log(a) P^\rho(a) U^* = \sum_{a \in \mathrm{spec}(\rho)} \log(a) U P^\rho(a) U^*,$$

where $UP^\rho(a)U^*$ are orthogonal projections. Therefore we get

$$\sum_{a \in \mathrm{spec}(\rho)} \log(a) U P^\rho(a) U^* = \sum_{a \in \mathrm{spec}(\rho)} \log(a) P^{U\rho U^*}(a) = \log(U\rho U^*).$$

□

Theorem 2.4. Let $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ and U unitary operators. Then

$$D(\rho \,||\, \sigma) = D(U\rho U^* \,||\, U\sigma U^*).$$

Proof. Using lemma (2.3) we get

$$\begin{aligned} D(U\rho U^* \,||\, U\sigma U^*) &= \mathrm{Tr}\big(U\rho U^*(\log(U\rho U^*) - \log(U\sigma U^*))\big) \\ &= \mathrm{Tr}\big(U\rho U^*(U\log(\rho)U^* - U\log(\sigma)U^*)\big) \\ &= \mathrm{Tr}\big(U\rho(\log(\rho) - \log(\sigma))U^*\big) \\ &= \mathrm{Tr}\big(\rho(\log(\rho) - \log(\sigma))\big) \\ &= D(\rho \,||\, \sigma), \end{aligned}$$

where in the penultimate line the cyclicity of the trace, and the definition of the U unitary matrices were used. □

2.3 Partial trace

Let $\rho_{AB}, \sigma_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$.We need to show that

$$D(\rho_{AB} \,||\, \sigma_{AB}) \geq D(\rho_A \,||\, \sigma_A),$$

where $\rho_A = \mathrm{Tr}_B \rho_{AB}$, $\sigma_A = \mathrm{Tr}_B \sigma_{AB}$.

Lemma 2.5. There exists a finite set of $\{U_j^B\}_{i=1}^r$ unitary operators on $\mathcal{B}(\mathcal{H}_B)$ and $\{p_i\}_{i=1}^r$ such that $p_i \geq 0$, $\sum_{i=1}^n p_i = 1$, so that for any $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ we have

$$\sum_{i=1}^r p_i (\mathrm{id}_A \otimes U_i^B) \rho_{AB} (\mathrm{id}_A \otimes U_i^B)^* = \frac{1}{\dim(B)} \rho_A \otimes \mathrm{id}_B, \qquad (\rho_A = \mathrm{Tr}_B \rho_{AB}).$$

Theorem 2.6. Let $\rho_{AB}, \sigma_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Then

$$D(\rho_A \,||\, \sigma_A) \leq D(\rho_{AB} \,||\, \sigma_{AB}),$$

where $\rho_A = \mathrm{Tr}_A \rho_{AB}$ and $\sigma_A = \mathrm{Tr}_A \sigma_{AB}$.

Proof. Using lemma (2.5) we have

$$\frac{1}{\dim(B)}\rho_A \otimes \mathrm{id}_B = \sum_{i=1}^{r} p_i (\mathrm{id}_A \otimes U_j^B)\rho_{AB}(\mathrm{id}_A \otimes U_j^B)^* =: \sum_{i=1}^{r} p_i \rho_{AB}^i, \tag{2.1}$$

$$\frac{1}{\dim(B)}\sigma_A \otimes \mathrm{id}_B = \sum_{i=1}^{r} p_i (\mathrm{id}_A \otimes U_j^B)\sigma_{AB}(\mathrm{id}_A \otimes U_j^B)^* =: \sum_{i=1}^{r} p_i \sigma_{AB}^i. \tag{2.2}$$

Then using (2.2.1) we can write

$$\begin{aligned}
D(\rho_A \,||\, \sigma_B) &\overset{(2.2.1)}{=} D(\rho_A \otimes \mathrm{id}_B \,||\, \sigma_B \otimes \mathrm{id}_B) \\
&\overset{(2.1)}{=} D(\sum_{i=1}^{r} p_i \rho_{AB}^i \,||\, \sum_{i=1}^{r} p_i \sigma_{AB}^i) \\
&\overset{\text{joint convexity}}{\geq} \sum_{i=1}^{r} p_i D(\rho_{AB}^i \,||\, \sigma_{AB}^i) \\
&= \sum_{i=1}^{r} p_i D((\mathrm{id}_A \otimes U_j^B)\rho_{AB}(\mathrm{id}_A \otimes U_j^B)^* \,||\, (\mathrm{id}_A \otimes U_j^B)\sigma_{AB}(\mathrm{id}_A \otimes U_j^B)^*),
\end{aligned}$$

where we could use the result (2.4) to get

$$\begin{aligned}
\sum_{i=1}^{r} p_i D((\mathrm{id}_A \otimes U_j^B)\rho_{AB}(\mathrm{id}_A \otimes U_j^B)^* \,||\, (\mathrm{id}_A \otimes U_j^B)\sigma_{AB}(\mathrm{id}_A \otimes U_j^B)^*) &= \sum_{i=1}^{r} p_i D(\rho_{AB} \,||\, \sigma_{AB}) \\
&= D(\rho_{AB} \,||\, \sigma_{AB}).
\end{aligned}$$

□

2.4 Data processing

Theorem 2.7 (Stinespring dilation)**.** Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ a CPTP map and $\rho \in \mathcal{S}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Then there exists a Hilbert space $\mathcal{H}_E$, $v \in \mathcal{H}_E$ and a unitary operation $U \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_E)$ such that

$$\Phi(\rho) = \mathrm{Tr}_E U(\rho \otimes |v\rangle\langle v|)U^*,$$

where Tr_E is the partial trace on $\mathcal{H}_E$.

Theorem 2.8 (Data processing)**.** Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a CPTP map. Then

$$D(\Phi(\rho) \,||\, \Phi(\sigma)) \leq D(\rho \,||\, \sigma).$$

Proof. Using Stinespring dilation (2.7) we have

$$\begin{aligned}
D(\Phi(\rho) \,||\, \Phi(\sigma)) &= D(\mathrm{Tr}_E U(\rho \otimes |v\rangle\langle v|)U^* \,||\, \mathrm{Tr}_E U(\sigma \otimes |v\rangle\langle v|)U^*) \\
&\overset{(2.6)}{\leq} D(U(\rho \otimes |v\rangle\langle v|)U^* \,||\, U(\sigma \otimes |v\rangle\langle v|)U^*) \\
&\overset{(2.4)}{=} D(\rho \otimes |v\rangle\langle v| \,||\, \sigma \otimes |v\rangle\langle v|) \\
&\overset{(2.2.1)}{=} D(\rho \,||\, \sigma).
\end{aligned}$$

□

3 Subadditivity theorems

Theorem 3.1. Let $\mathcal{H}_{AB} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\rho_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$, and let $\rho_A = \mathrm{Tr}_B \rho_{AB}$ and $\rho_B = \mathrm{Tr}_A \rho_{AB}$. Then the subadditivity inequality

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B) \tag{3.1}$$

holds.

Theorem 3.2. In a more complex situation, we have a stronger inequality, called the strong subadditivity (SSA):

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}).$$

Proof. Using (2.6) we have

$$D(\rho_{ABC} \,||\, \rho_A \otimes \rho_{BC}) - D(\rho_{AB} \,||\, \rho_A \otimes \rho_B) \geq 0,$$

with the CPTP map $\Phi := \mathrm{id}_{AB} \otimes \mathrm{Tr}_C$. Using the definition (1.2) we get

$$\begin{aligned} &D(\rho_{ABC} \,||\, \rho_A \otimes \rho_{BC}) - D(\rho_{AB} \,||\, \rho_A \otimes \rho_B) \\ &\quad = -S(\rho_{ABC}) + S(\rho_A) + S(\rho_{BC}) + S(\rho_{AB}) - S(\rho_A) - S(\rho_B) \\ &\quad = -S(\rho_{ABC}) + S(\rho_{BC}) + S(\rho_{AB}) - S(\rho_B) \geq 0, \end{aligned}$$

which is just the desired inequality. □

Why is theorem 3.2 strong? Consider the case when $\dim \mathcal{H}_2 = 1$. Then it leads to the regular subadditivity relation in (3.1).

4 Fundamental inequalities

Theorem 4.1 (Klein's inequality)**.** For $A, B \in \mathcal{B}(\mathcal{H})_+$, we have the inequality

$$\mathrm{Tr}\ A(\log A - \log B) \geq \mathrm{Tr}\ (A - B)$$

with equality if and only if $A = B$.

Theorem 4.2 (Golden-Thompson inequality)**.** Let $A, B \in \mathcal{B}(\mathcal{H})$. Then

$$\mathrm{Tr}\ e^{A+B} \leq \mathrm{Tr}\ e^A e^B.$$

But an analogous statement does not hold for three operators.

Proposition 4.1. For any $K \in \mathcal{B}(\mathcal{H})_{sa}$, the following function is concave for $A > 0$:

$$F : A \mapsto \mathrm{Tr} \exp(K + \log A).$$

Theorem 4.3 (Lieb's triple matrix inequality)**.** For any $R, S, T > 0$

$$\mathrm{Tr}\, e^{\log R - \log S + \log T} \leq \mathrm{Tr} \int_0^\infty R \frac{1}{S + uI} T \frac{1}{S + uI} du.$$

Proof. If $F(A)$ is concave and homogeneous, i. e. $F(xA) = xF(A)$, then

$$\lim_{x \to 0} \frac{F(A + xB) - F(A)}{x} \geq F(B).$$

Applying this with (4.1) setting $A = S$, $B = T$, $K = \log R - \log S$, we get

$$\mathrm{Tr} \exp(\log R - \log S + \log T) \leq \lim_{x \to 0} \frac{\mathrm{Tr} \exp\big(\log R - \log S + \log(S + xT)\big) - \mathrm{Tr} R}{x}. \tag{4.1}$$

Here we can use

$$\log(S + xT) - \log S = \int_0^\infty \frac{1}{S + uI} xT \frac{1}{S + xT + uI} du.$$

We can rewrite the numerator in the right hand side of (4.1) to get

$$\begin{aligned}
&\mathrm{Tr}\exp(\log R - \log S + \log(S + xT) - \mathrm{Tr}R \\
&= \mathrm{Tr}\exp\left(\log R + x\int_0^\infty \frac{1}{S+uI}T\frac{1}{S+xT+uI}du\right) - \mathrm{Tr}R \\
&= x\mathrm{Tr}R\int_0^\infty \frac{1}{S+uI}T\frac{1}{S+xT+uI}du + \mathcal{O}(x^2),
\end{aligned} \tag{4.2}$$

where in the last line we expanded in x. Using (4.2), we get

$$\mathrm{Tr}\exp(\log R - \log S + \log T) \leq \mathrm{Tr}\left(R\int_0^\infty \frac{1}{S+uI}T\frac{1}{S+xT+uI}du\right),$$

which is the inequality we expected. □

5 Subadditivity proofs

Lemma 5.1. Let $A \in \mathcal{B}(\mathcal{H}_A)$. Then the following is true:

$$\exp\big((\log A)\otimes \mathrm{id}_B\big) = A \otimes \mathrm{id}_B,$$

where id_B is the identity of the linear operators on some Hilbert space $\mathcal{B}(\mathcal{H}_B)$.

Proof. Using the spectral decomposition (1.1), we have

$$\exp\big((\log A)\otimes \mathrm{id}_B\big) = \exp\left(\sum_{a\in\mathrm{spec}(A)} \log(a)P^A(a)\otimes \mathrm{id}_B\right),$$

where we note that $P^A(a)\otimes \mathrm{id}_B$ is an orthonormal projection (as $P^A(a)$), so we could write

$$\begin{aligned}
\exp\left(\sum_{a\in\mathrm{spec}(A)} \log(a)P^A(a)\otimes \mathrm{id}_B\right) &= \sum_{a\in\mathrm{spec}(A)} (\exp\circ\log)(a)P^A(a)\otimes \mathrm{id}_B \\
&= \sum_{a\in\mathrm{spec}(A)} aP^A(a)\otimes \mathrm{id}_B \\
&= A\otimes \mathrm{id}_B.
\end{aligned}$$

□

Lemma 5.2. Let $R \in \mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B)$, $A \in \mathcal{B}(\mathcal{H}_A)$. Then the following is true:

$$\log R = (\log A)\otimes \mathrm{id}_B \implies R = A\otimes \mathrm{id}_B.$$

Proof. Simply using the lemma (5.1) we have

$$R = \exp(\log R) = \exp\big((\log A)\otimes \mathrm{id}_B\big) = A\otimes \mathrm{id}_B.$$

□

Theorem 5.3. Let $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_A\otimes\mathcal{H}_B\otimes\mathcal{H}_C)$. Then

$$S(\rho_{ABC}) \leq S(\rho_{AB}) + S(\rho_{BC}).$$

Proof. Using Klein's inequality (4.1), choosing $A = \rho_{ABC}$ and $\log B = (\log\rho_{AB}\otimes \mathrm{id}_C + \mathrm{id}_A\otimes\log\rho_{BC})$ we get

$$-S(\rho_{ABC}) + S(\rho_{AB}) + S(\rho_{BC}) \geq 1 - \mathrm{Tr}\exp(\log\rho_{AB}\otimes \mathrm{id}_C + \mathrm{id}_A\otimes\log\rho_{BC}).$$

Using then the Golden-Thomson inequality (4.2) for the right hand side we obtain

$$1 - \operatorname{Tr}\exp(\log\rho_{AB}\otimes\mathrm{id}_C + \mathrm{id}_A\otimes\log\rho_{BC}) \geq 1 - \operatorname{Tr}\big(\exp(\log\rho_{AB}\otimes\mathrm{id}_C)\exp(\mathrm{id}_A\otimes\log\rho_{BC})\big).$$

Using the lemma (5.1), we have

$$\exp(\log\rho_{AB}\otimes\mathrm{id}_C) = \rho_{AB}\otimes\mathrm{id}_C$$

Similarly for the other term in the exponent we can write

$$\exp(\mathrm{id}_A\otimes\log\rho_{BC}) = \mathrm{id}_A\otimes\rho_{BC}.$$

Now we have

$$\begin{aligned}
-S(\rho_{ABC}) + S(\rho_{AB}) + S(\rho_{BC}) &\geq 1 - \operatorname{Tr}\big(\rho_{AB}\otimes\mathrm{id}_C\big)\big(\mathrm{id}_A\otimes\rho_{BC}\big)\\
&= 1 - \big(\operatorname{Tr}_B\circ\operatorname{Tr}_A\circ\operatorname{Tr}_C\big)\big(\rho_{AB}\otimes\mathrm{id}_C\big)\big(\mathrm{id}_A\otimes\rho_{BC}\big)\\
&= 1 - \big(\operatorname{Tr}_B\circ\operatorname{Tr}_A\big)\big(\rho_{AB}\big)\big(\mathrm{id}_A\otimes\rho_B\big)\\
&= 1 - \operatorname{Tr}_B\rho_B^2.
\end{aligned}$$

Where we could use the inequality

$$\operatorname{Tr}_B\rho_B \geq \operatorname{Tr}_B\rho_B\rho_B,$$

since $\rho_B \leq \mathrm{id}_B$. Therefore we have

$$\begin{aligned}
-S(\rho_{ABC}) + S(\rho_{AB}) + S(\rho_{BC}) &\geq 1 - \operatorname{Tr}_B\rho_B\rho_B \geq 1 - \operatorname{Tr}_B\rho_B = 0\\
\implies S(\rho_{AB}) + S(\rho_{BC}) &\geq S(\rho_{ABC}).
\end{aligned}$$

□

Lemma 5.4. Let $A \in \mathcal{B}(\mathcal{H}_A)$ and $u \in \mathbb{R}$. Then the following is true:

$$(A\otimes\mathrm{id}_B + u\,\mathrm{id}_{AB})^{-1} = (A + u\,\mathrm{id}_A)^{-1}\otimes\mathrm{id}_B,$$

where $\mathrm{id}_{AB} = \mathrm{id}_A\otimes\mathrm{id}_B$, and id_B is the identity of the linear operators on some Hilbert space $\mathcal{B}(\mathcal{H}_B)$.

Proof. Using the spectral decomposition (1.1)

$$\begin{aligned}
(A\otimes\mathrm{id}_B + u\,\mathrm{id}_{AB})^{-1} &= \left(\sum_{a\in\mathrm{spec}(A)} aP^A(a)\otimes\mathrm{id}_B + u\,\mathrm{id}_A\otimes\mathrm{id}_B\right)^{-1}\\
&= \left(\sum_{a\in\mathrm{spec}(A)} aP^A(a)\otimes\mathrm{id}_B + u\,\mathrm{id}_A\otimes\mathrm{id}_B\right)^{-1},
\end{aligned}$$

then we can use $\mathrm{id}_A = \sum_{a\in\mathrm{spec}(A)} P^A(a)$ assuming a full rank, therefore

$$\begin{aligned}
\left(\sum_{a\in\mathrm{spec}(A)} aP^A(a)\otimes\mathrm{id}_B + u\,\mathrm{id}_A\otimes\mathrm{id}_B\right)^{-1} &= \left(\sum_{a\in\mathrm{spec}(A)} (a+u)P^A(a)\otimes\mathrm{id}_B\right)^{-1}\\
&= \sum_{a\in\mathrm{spec}(A)} (a+u)^{-1}P^A(a)\otimes\mathrm{id}_B\\
&= (A + u\,\mathrm{id}_A)^{-1}\otimes\mathrm{id}_B,
\end{aligned}$$

□

Theorem 5.5 (SSA)**.**

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}).$$

Proof. Using Klein's inequality (4.1), and setting $A = \rho_{ABC}$ and $\log B = \log\rho_{AB}\otimes\mathrm{id}_C - \mathrm{id}_A\otimes\log\rho_B\otimes\mathrm{id}_C +$

$\mathrm{id}_A \otimes \log \rho_{BC}$

$$- S(\rho_{ABC}) - S(\rho_B) + S(\rho_{AB}) + S(\rho_{BC}) \geq \mathrm{Tr}\big(\rho_{ABC} - \exp(\log \rho_{AB} \otimes \mathrm{id}_C - \mathrm{id}_A \otimes \log \rho_B \otimes \mathrm{id}_C + \mathrm{id}_A \otimes \log \rho_{BC})\big).$$

Convenient choice to use the same notations that was used at Lieb's theorem (4.3) as a preparation:

$$\begin{aligned} \log R = \log \rho_{AB} \otimes \mathrm{id}_C &\implies R = \rho_{AB} \otimes \mathrm{id}_C \\ \log S = \mathrm{id}_A \otimes \log \rho_B \otimes \mathrm{id}_C &\implies S = \mathrm{id}_A \otimes \rho_B \otimes \mathrm{id}_C \\ \log T = \mathrm{id}_A \otimes \log \rho_{BC} &\implies T = \mathrm{id}_A \otimes \rho_{BC}. \end{aligned}$$

Then, applying Lieb's theorem (4.3) for the right hand side, we obtain

$$\begin{aligned} &- S(\rho_{ABC}) - S(\rho_B) + S(\rho_{AB}) + S(\rho_{BC}) \\ &\quad \geq \mathrm{Tr}\left[\rho_{ABC} - \int_0^\infty (\rho_{AB} \otimes \mathrm{id}_C) \frac{1}{\mathrm{id}_A \otimes \rho_B \otimes \mathrm{id}_C + u\, \mathrm{id}_{ABC}} (\mathrm{id}_A \otimes \rho_{BC}) \frac{1}{\mathrm{id}_A \otimes \rho_B \otimes \mathrm{id}_C + u\, \mathrm{id}_{ABC}} du\right]. \end{aligned}$$

Here we can use the lemma (5.4):

$$\begin{aligned} &\mathrm{Tr}\left[\rho_{ABC} - \int_0^\infty (\rho_{AB} \otimes \mathrm{id}_C) \frac{1}{\mathrm{id}_A \otimes \rho_B \otimes \mathrm{id}_C + u\, \mathrm{id}_{ABC}} (\mathrm{id}_A \otimes \rho_{BC}) \frac{1}{\mathrm{id}_A \otimes \rho_B \otimes \mathrm{id}_C + u\, \mathrm{id}_{ABC}} du\right] \\ &= \mathrm{Tr}\left[\rho_{ABC} - \int_0^\infty (\rho_{AB} \otimes \mathrm{id}_C) \left(\mathrm{id}_A \otimes \frac{1}{\rho_B + u\, \mathrm{id}_B} \otimes \mathrm{id}_C\right) (\mathrm{id}_A \otimes \rho_{BC}) \left(\mathrm{id}_A \otimes \frac{1}{\rho_B + u\, \mathrm{id}_B} \otimes \mathrm{id}_C\right) du\right] \\ &= \mathrm{Tr}\rho_{ABC} - (\mathrm{Tr}_B \circ \mathrm{Tr}_A \circ \mathrm{Tr}_C) \\ &\int_0^\infty (\rho_{AB} \otimes \mathrm{id}_C) \left(\mathrm{id}_A \otimes \frac{1}{\rho_B + u\, \mathrm{id}_B} \otimes \mathrm{id}_C\right) (\mathrm{id}_A \otimes \rho_{BC}) \left(\mathrm{id}_A \otimes \frac{1}{\rho_B + u\, \mathrm{id}_B} \otimes \mathrm{id}_C\right) du \\ &= \mathrm{Tr}\rho_{ABC} - (\mathrm{Tr}_B \circ \mathrm{Tr}_A) \int_0^\infty \rho_{AB} \left(\mathrm{id}_A \otimes \frac{1}{\rho_B + u\, \mathrm{id}_B}\right) (\mathrm{id}_A \otimes \rho_B) \left(\mathrm{id}_A \otimes \frac{1}{\rho_B + u\, \mathrm{id}_B}\right) du \\ &= \mathrm{Tr}\rho_{ABC} - \mathrm{Tr}_B \int_0^\infty \rho_B \left(\frac{1}{\rho_B + u\, \mathrm{id}_B}\right) \rho_B \left(\frac{1}{\rho_B + u\, \mathrm{id}_B}\right) du. \end{aligned}$$

Lieb's triple matrix inequality (4.3) holds for commuting operators, because they can be diagonalizable in the same basis. ρ_B trivially commutes with itself, therefore Lieb's triple matrix inequality (4.3) is an equality for this case. Showing it explicitly:

$$\begin{aligned} \mathrm{Tr}_B \int_0^\infty \rho_B \left(\frac{1}{\rho_B + u\, \mathrm{id}_B}\right) \rho_B \left(\frac{1}{\rho_B + u\, \mathrm{id}_B}\right) du &= \mathrm{Tr}_B \sum_{b \in \mathrm{spec}(\rho_B)} \int_0^\infty b\left(\frac{1}{b+u}\right) b\left(\frac{1}{b+u}\right) du P^{\rho_B}(b) \\ &= \mathrm{Tr}_B \sum_{b \in \mathrm{spec}(\rho_B)} b^2 \int_0^\infty \left(\frac{1}{b+u}\right)^2 du P^{\rho_B}(b) \\ &= \mathrm{Tr}_B \sum_{b \in \mathrm{spec}(\rho_B)} b^2 \frac{1}{b} P^{\rho_B}(b) \\ &= \mathrm{Tr}_B \sum_{b \in \mathrm{spec}(\rho_B)} b P^{\rho_B}(b) \\ &= \mathrm{Tr}_B \rho_B. \end{aligned}$$

Finally, we conclude that

$$\mathrm{Tr}\rho_{ABC} - \mathrm{Tr}_B \int_0^\infty \rho_B \left(\frac{1}{\rho_B + u\, \mathrm{id}_B}\right) \rho_B \left(\frac{1}{\rho_B + u\, \mathrm{id}_B}\right) du = \mathrm{Tr}\rho_{ABC} - \mathrm{Tr}_B \rho_B = 0.$$

□

References

[1] M. B. Ruskai. Inequalities for quantum entropy: A Review with conditions for equality J. Math. Phys. 43, 4358–4375 (2002); Erratum. J. Math. Phys. 46, 019901 (2005).